

Tightness results for infinite-slit limits of the chordal Loewner equation

Andrea del Monaco

Ikkei Hotta*

Sebastian Schleißinger†

August 16, 2016

Abstract

In this note we consider a multi-slit Loewner equation with constant coefficients that describes the growth of multiple SLE curves connecting N points on \mathbb{R} to infinity within the upper half-plane. For every $N \in \mathbb{N}$, this equation provides a measure valued process $t \mapsto \{\alpha_{N,t}\}$, and we are interested in the limit behaviour as $N \rightarrow \infty$. We prove tightness of the sequence $\{\alpha_{N,t}\}_{N \in \mathbb{N}}$ under certain assumptions and address some further problems.

Keywords: chordal Loewner equation, stochastic Loewner evolution, multiple SLE, complex Burgers equation, tightness, quadratic differentials

2010 Mathematics Subject Classification: 60J67, 37L05.

Contents

1	Introduction	1
2	Tightness of a multiple SLE process	2
2.1	Geometry and Loewner Theory	2
2.2	Single and Multiple SLE	3
2.3	The chordal Loewner equation for $H_{\mathbb{H},\kappa}((x_1, \dots, x_N), \infty)$	4
2.4	Tightness	5
2.5	The simultaneous case	8
2.6	Examples	9
2.7	Problems and Remarks	12
3	Trajectories of a certain quadratic differential	13

1 Introduction

In [dMS16], the second and third author noted that the conformal mappings for a certain multiple SLE (Schramm-Loewner evolution) process for N simple curves in the upper half-plane \mathbb{H} converges as $N \rightarrow \infty$. The deterministic limit has a simple description: The conformal mappings $f_t : \mathbb{H} \rightarrow \mathbb{H}$ satisfy the Loewner PDE

$$\frac{\partial f_t(z)}{\partial t} = -\frac{\partial f_t(z)}{\partial z} \cdot M_t(z), \quad f_0(z) = z \in \mathbb{H},$$

where M_t satisfies the complex Burgers equation

$$\frac{\partial M_t(z)}{\partial t} = -2 \frac{\partial M_t(z)}{\partial z} \cdot M_t(z),$$

see Section 2.5 for more details. In several situations, partial differential equations of this type appear to describe the limit of N -particle systems; see [Cha92, RS93, CL97].

*Supported by the JSPS KAKENHI Grant no. 26800053.

†Supported by the ERC grant “HEVO - Holomorphic Evolution Equations” no. 277691.

In Section 2, we consider again the same multiple SLE measure for N curves connecting N points on \mathbb{R} with ∞ . We describe the growth of these curves by a Loewner equation with weights that correspond to the speed for these curves in the growth process, and we obtain an abstract differential equation for limit points as $N \rightarrow \infty$ (Corollary 2.7).

Furthermore, in Section 3 we see that an equation of a similar type also appears in the limit behaviour of a Loewner equation describing the growth of trajectories of a certain quadratic differential.

2 Tightness of a multiple SLE process

2.1 Geometry and Loewner Theory

In this section we briefly recall the general background of hulls in the upper half-plane and the chordal Loewner equation.

A domain $D \subsetneq \hat{\mathbb{C}}$ is said to be a *Jordan domain* if ∂D is homeomorphically equivalent to the unit circle $\mathbb{T} = \partial\mathbb{D}$. Let Γ be a subset of \overline{D} such that there exist some $T > 0$ and a homeomorphism $\gamma: [0, T] \rightarrow \Gamma$ with $\gamma(0, T) \subset D$ and $\gamma(0) \in \partial D$. Then, if $\gamma(T) \in D$, the set $\Gamma \cap D = \Gamma \setminus \gamma(0)$ is said to be a *slit* in D , and if $\gamma(T) \in \partial D$ as well, Γ is referred to as a *chord* (in D).

Since by the Riemann Mapping Theorem (see, e.g., [Pom75, Section 1.1]) D is conformally equivalent to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$, it suffices to consider the case $D = \mathbb{H}$ and $\gamma(0) \in \mathbb{R}$. In this setting, in particular, one may also introduce the more general notion of *hull*, i.e. a subset $A \subset \mathbb{H}$ such that $\overline{A} \cap \mathbb{H} = A$ and $\mathbb{H} \setminus A$ is simply connected.

It is known that if $A \subset \mathbb{H}$ is a bounded hull, then there exists a unique conformal mapping $g_A: \mathbb{H} \setminus A \rightarrow \mathbb{H}$ with *hydrodynamic normalization* (see [Law05, Proposition 3.34]), meaning that

$$g_A(z) = z + \frac{b}{z} + \tilde{g}(z) \quad \text{as } z \rightarrow \infty$$

for a holomorphic function \tilde{g} with $\angle \lim_{z \rightarrow \infty} z \cdot \tilde{g}(z) = 0$.

The quantity $b = \text{hcap}(A) \geq 0$ is called the *half-plane capacity* of A .

The mapping g_A can be embedded into the solution of a Loewner equation as follows. Let $T > 0$ be defined by $2T = \text{hcap}(A)$. Then there exists a family $\{\mu_t\}_{t \in [0, T]}$ of probability measures on \mathbb{R} , with the property that $t \mapsto \int_{\mathbb{R}} \frac{1}{z-u} \mu_t(du)$ is measurable for every $z \in \mathbb{H}$, such that the solution $\{g_t\}_{t \in [0, T]}$ of the chordal Loewner equation

$$\begin{cases} \frac{dg_t(z)}{dt} = \int_{\mathbb{R}} \frac{2}{g_t(z) - u} \mu_t(du) & \text{for almost every } t \in [0, T] \\ g_0(z) = z \in \mathbb{H} \end{cases} \quad (2.1)$$

satisfies $g_A = g_T$. This follows from [GB92, Theorem 5] and considering the time-reversed flow and the inverse mapping g_A^{-1} .

Conversely, one can always solve (2.1) and obtain conformal mappings with hydrodynamic normalization; see [GB92, Theorem 4] or [Law05, Theorem 4.5].

For $z \in \mathbb{H}$ fixed, the solution $t \mapsto g_t(z)$ of (2.1) may have a finite lifetime $T(z) > 0$, namely $g_t(z) \in \mathbb{H}$ for all $t < T(z)$ and $\text{Im}(g_t(z)) \rightarrow 0$ as $t \uparrow T(z)$.

If we fix a time $t > 0$ and let $K_t = \{z \in \mathbb{H} \mid T(z) \leq t\}$, then K_t is a (not necessarily bounded) hull and the mapping $z \mapsto g_t(z)$ is the conformal mapping from $\mathbb{H} \setminus K_t$ onto \mathbb{H} with hydrodynamic normalization. Furthermore, the hulls K_t are strictly growing, i.e. $K_s \subsetneq K_t$ whenever $s < t$, and $\text{hcap}(K_t) = 2t$.

When the hull A is a slit Γ , equation (2.1) necessarily has the form

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - U(t)}, \quad g_0(z) = z \in \mathbb{H}, \quad (2.2)$$

with a unique, continuous driving function $U: [0, T] \rightarrow \mathbb{R}$ (see [dMG16], and the references therein, for more details). In this case, we obtain a parametrization γ of Γ by setting $\gamma(0, t] = K_t$, which is equivalent to requiring $\text{hcap}(\gamma(0, t]) = 2t$. We call γ the *parametrization by half-plane capacity* of Γ .

More generally, if A is the union of n slits $\Gamma_1, \dots, \Gamma_n$ with pairwise disjoint closures, i.e. $\overline{\Gamma_j} \cap \overline{\Gamma_k} = \emptyset$ whenever $j \neq k$, then (2.1) must have the form

$$\frac{dg_t(z)}{dt} = \sum_{j=1}^n \frac{2\lambda_j(t)}{g_t(z) - U_j(t)}, \quad g_0(z) = z \in \mathbb{H}, \quad (2.3)$$

where $U_j : [0, T] \rightarrow \mathbb{R}$ are continuous and $\lambda_j : [0, T] \rightarrow [0, 1]$ are measurable functions with $\sum_{j=1}^n \lambda_j(t) = 1$ for every t ; see [Boe15, Theorem 2.54]. In this way, we obtain parametrizations $\gamma_1, \dots, \gamma_n$ of $\Gamma_1, \dots, \Gamma_n$ by requiring $K_t = \cup_{j=1}^n \gamma_j(0, t]$.

It is worth noting that, for $n > 1$, a representation of A by (2.3) is not unique. For example, we could first generate slit Γ_k only, i.e. $\lambda_k(t) = 1 = 1 - \lambda_j(t)$ for $j \neq k$ and t small enough.

Remark 2.1. *The coefficients $\lambda_j(t)$ can be thought of as the speed of growth of the slit Γ_j at time t . More precisely, we have the following relation:*

Fix j and $t_0 \geq 0$, assume that g_t is differentiable at $t = t_0$ and consider the curve $\tilde{\gamma}(h) = g_{t_0}(\gamma_j[t_0, t_0 + h])$. Let $b(h) := \text{hcap}(\tilde{\gamma}(0, h])$ be the half-plane capacity of the slit $\tilde{\gamma}(0, h]$. Then $b(h)$ is differentiable at $h = 0$ with $b'(0) = \lambda_j(t_0)$, see [Boe15, Theorem 2.36].

2.2 Single and Multiple SLE

In what follows, $\kappa \in (0, 4]$ is a fixed parameter and $D \subsetneq \mathbb{C}$ is a Jordan domain.

Fix two points $x, y \in \partial D$ and assume that ∂D is analytic in neighbourhoods of x and y .

The chordal Schramm-Loewner evolution (SLE) of a random curve $\Gamma \subset D$ for the data D, x, y, κ can be viewed as a certain probability measure $\mu_{D, \kappa}(x, y)$ on the space of all chords connecting the points x and y within D . As one property of SLE is conformal invariance, it suffices to describe the SLE when $D = \mathbb{H}$, $x = 0$, and $y = \infty$. In this setting, the evolution of Γ can be described efficiently as follows. Let γ be a parametrization of Γ with $\gamma(0) = 0$ and assume that $\gamma[0, T]$ is parametrized by half-plane capacity for every $T > 0$. The random conformal mapping $g_t := g_{\gamma(0, t]}$ then satisfies the Loewner equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z, \quad (2.4)$$

where B_t is a standard one-dimensional Brownian motion.

Notice that one may also consider SLE for $\kappa > 4$. But then the measure is no longer supported on simple curves, and we are not interested in such a case here. For further information and a thorough treatment of SLE we refer to [Law05].

Next, we describe multiple SLE as it was introduced in [KL07].

Let $N \in \mathbb{N}$ and fix $2N$ pairwise distinct points $p_1, \dots, p_{2N} \in \partial D$ in counter-clockwise order. Assume that ∂D is analytic in a neighbourhood of p_k , $k = 1, \dots, 2N$.

We call the pair (\mathbf{x}, \mathbf{y}) of two tuples $\mathbf{x} = (x_1, \dots, x_N)$, $\mathbf{y} = (y_1, \dots, y_N)$ a *configuration* for these points if

- a) $\{x_1, \dots, x_N, y_1, \dots, y_N\} = \{p_1, \dots, p_{2N}\}$,
- b) there exist N pairwise disjoint chords γ_k connecting x_k to y_k within D , $k = 1, \dots, N$,
- c) $x_1 = p_1$ and x_1, x_2, \dots, x_N , as well as x_1, x_k, y_k , for every $k \geq 2$, are in counter-clockwise order.

The points in \mathbf{x} can be thought of as the starting points of these chords. Then \mathbf{y} represents the end points and the assumption in c) just prevents us from getting a new configuration by exchanging a starting point of one curve with its endpoint. A simple combinatorial calculation gives that there exist

$$C_N = \frac{(2N)!}{(N+1)! N!}$$

many configurations for $2N$ points.

Fix now a configuration (\mathbf{x}, \mathbf{y}) . The *configurational multiple SLE* $Q_{D, \kappa}(\mathbf{x}, \mathbf{y})$ is a positive finite measure on the space of all N -tuples $(\gamma_1, \dots, \gamma_N)$, where γ_k is a chord in D connecting x_k and y_k and $\gamma_k \cap \gamma_j = \emptyset$ whenever $j \neq k$. One may construct the $Q_{D, \kappa}(\mathbf{x}, \mathbf{y})$ by means of the Brownian loop measure (see [KL07] for details).

If we let $H_{D, \kappa}(\mathbf{x}, \mathbf{y})$ be the mass of $Q_{D, \kappa}(\mathbf{x}, \mathbf{y})$, then we can write

$$Q_{D, \kappa}(\mathbf{x}, \mathbf{y}) = H_{D, \kappa}(\mathbf{x}, \mathbf{y}) \cdot \mu_{D, \kappa}(\mathbf{x}, \mathbf{y})$$

for some probability measure $\mu_{D,\kappa}(\mathbf{x}, \mathbf{y})$.

Thus, one may view $Q_{D,\kappa}(\mathbf{x}, \mathbf{y})$ as a probability measure for the underlying configuration with weight $H_{D,\kappa}(\mathbf{x}, \mathbf{y})$. Then we may use such weights as partition functions to combine multiple SLE for different configurations. Namely, if $\mathbf{p} = (p_1, \dots, p_{2N})$ and $S(\mathbf{p})$ is the set of all configurations, then the probability for $(\mathbf{x}, \mathbf{y}) \in S(\mathbf{p})$ will be given by

$$p(\mathbf{x}, \mathbf{y}) = \frac{H_{D,\kappa}(\mathbf{x}, \mathbf{y})}{\sum_{(\mathbf{v}, \mathbf{w}) \in S(\mathbf{p})} H_{D,\kappa}(\mathbf{v}, \mathbf{w})}. \quad (2.5)$$

Example 2.2. Consider the case $N = 2$ and $\kappa = 3$. Then there are two possible configurations C_1 and C_2 , and $\mu_{D,3}(\{C_1, C_2\})$ describes the scaling limit for the Ising model with corresponding boundary conditions (see [Koz09]). The probability p for obtaining configuration C_1 is given by

$$p = \frac{H_{D,3}(C_1)}{H_{D,3}(C_1) + H_{D,3}(C_2)}.$$

On the other hand, $H_{D,\kappa}(\mathbf{x}, \mathbf{y})$ may also be used to write down a Loewner equation that governs the growth of multiple SLE curves, see [KL07, Section 4].

Again, because of conformal invariance, it suffices just to consider the case $D = \mathbb{H}$, where $p_1, \dots, p_{2N} \in \mathbb{R} \cup \{\infty\}$. In this setting, the number $H_{\mathbb{H},\kappa}(\mathbf{x}, \mathbf{y})$ is known explicitly only for some special cases:

- (i) for $N = 1$ and $(x, y) = (0, \infty)$, one simply takes $H_{\mathbb{H},\kappa}(0, \infty) = 1$ as a definition, which would then yield $Q_{D,\kappa} = \mu_{\mathbb{H},\kappa}$, i.e. the chordal SLE probability measure as described in 2.4;
- (ii) if $N = 1$ and $x, y \in \mathbb{R}$, then $H_{\mathbb{H},\kappa}(x, y) = |y - x|^{-2b}$, $b = \frac{6-\kappa}{2\kappa}$;
- (iii) a special case for $\kappa = 2$ is given in [KL07] (see Remark after Proposition 3.3);
- (iv) for $N = 2$, $H_{\mathbb{H},\kappa}((x_1, x_2), (y_1, y_2))$ can be expressed by a formula involving hypergeometric functions (see [KL07, Proposition 3.4]).

We point out that multiple SLE can also be approached by requiring certain properties for the multi-slit Loewner equation, which leads to local properties of $H_{D,\kappa}(\mathbf{x}, \mathbf{y})$ as a partition function. A framework for describing $H_{D,\kappa}(\mathbf{x}, \mathbf{y})$ as the solution to certain differential equations is discussed in the recent works [FK15a, FK15b, FK15c, FK15d, KP15]. We also refer to the articles [Car03, BBK05, Gra07, Dub07].

Remark 2.3. Notice that one may consider $Q_{\mathbb{H},\kappa}(\mathbf{x}, \mathbf{y})$ also for a configuration where $y_j = y_k$ (or $x_j = x_k$, or both) for certain $j \neq k$. This is done by considering the disjoint case $y_j \neq y_k$ first and then taking a scaled limit.

In particular, if $(x_1, \dots, x_N) = (\infty, \dots, \infty)$, then one has

$$H_{\mathbb{H},\kappa}((x_1, \dots, x_N), \infty) := H_{\mathbb{H},\kappa}((x_1, \dots, x_N), (\infty, \dots, \infty)) := \prod_{1 \leq j < k \leq N} (x_k - x_j)^{2/\kappa}. \quad (2.6)$$

See [BBK05, Section 4.6], and the references therein, for more details.

2.3 The chordal Loewner equation for $H_{\mathbb{H},\kappa}((x_1, \dots, x_N), \infty)$

Let $N \in \mathbb{N}$ and $x_{N,1} < \dots < x_{N,N}$ be N points on \mathbb{R} . The growth of N random curves from $\mu_{\mathbb{H},\kappa}((x_{N,1}, \dots, x_{N,N}), \infty)$ can be described by a Loewner equation as follows:

First, choose $\lambda_{N,1}, \dots, \lambda_{N,N} \in (0, 1)$ such that $\sum_{k=1}^N \lambda_{N,k} = 1$.

Next, we define N random processes $V_{N,1}, \dots, V_{N,N}$ on \mathbb{R} as the solution of the SDE system

$$dV_{N,k}(t) = \sum_{j \neq k} \frac{2(\lambda_{N,k} + \lambda_{N,j})}{V_{N,k}(t) - V_{N,j}(t)} dt + \sqrt{\kappa \lambda_{N,k}} dB_{N,k}(t), \quad V_{N,k}(0) = x_{N,k}, \quad (2.7)$$

where $B_{N,1}, \dots, B_{N,N}$ are N independent standard Brownian motions and $\kappa \in [0, 4]$. Although multiple SLE was only defined for $\kappa \in (0, 4]$, in this particular case one may also consider the deterministic case $\kappa = 0$.

The corresponding N -slit Loewner equation

$$\frac{d}{dt} g_{N,t}(z) = \sum_{k=1}^N \frac{2\lambda_{N,k}}{g_{N,t}(z) - V_{N,k}(t)}, \quad g_{N,0}(z) = z \in \mathbb{H}, \quad (2.8)$$

describes the growth of N multiple SLE curves growing from $x_{N,1}, \dots, x_{N,N}$ to ∞ ; see [BBK05], p. 1130 (where the function Z is the partition function (2.6), see equation (4) on p. 1138). The function $z \mapsto g_{N,t}(z)$ is the conformal mapping $g_{\gamma_{N,1}[0,t] \cup \dots \cup \gamma_{N,N}[0,t]}$ for N random simple curves $\gamma_{N,k} : [0, \infty) \rightarrow \mathbb{H}$, which are non-intersecting and $\gamma_{N,k}(0) = x_{N,k}$.

We are interested in the limit $N \rightarrow \infty$ of the growing curves, i.e. the convergence of $\gamma_{N,1}[0, t] \cup \dots \cup \gamma_{N,N}[0, t]$ to a hull K_t . To be more precise, we would like to answer the following question once that some $t > 0$ has been fixed: under which conditions does the sequence $\mathbb{H} \setminus (\gamma_{N,1}[0, t] \cup \dots \cup \gamma_{N,N}[0, t])$ of domains converge to a (simply connected) domain $\mathbb{H} \setminus K_t$ with respect to kernel convergence (check Figure 1)?

According to Carathéodory's Kernel Theorem (Theorem 1.8 in [Pom75]), the above question is equivalent to asking for locally uniform convergence of the mappings $g_{N,t}$ to a conformal mapping $g_t : \mathbb{H} \setminus K_t \rightarrow \mathbb{H}$. Also, we would like to be able to describe g_t again by a Loewner equation.

Let δ_x be the point measure in x with mass 1 and define

$$\alpha_{N,t} = \sum_{k=1}^N \lambda_{N,k} \delta_{V_{N,k}(t)}. \quad (2.9)$$

Then equation (2.8) can be written as

$$\frac{d}{dt} g_{N,t} = \int_{\mathbb{R}} \frac{2}{g_{N,t} - u} d\alpha_{N,t}(u). \quad (2.10)$$

Assume now that

$$\alpha_{N,0} \xrightarrow{\mathbf{w}} \alpha \quad \text{as } N \rightarrow \infty, \quad (2.11)$$

where we denoted with “ $\xrightarrow{\mathbf{w}}$ ” the weak convergence and where α is again a probability measure. We wish to know whether the sequence $\{\alpha_{N,t}\}_{N \in \mathbb{N}}$ of stochastic measure-valued processes converges. In what follows, we show that, under certain assumptions for $x_{N,k}$ and $\lambda_{N,k}$, this sequence is tight and that each limit process satisfies the same differential equation.



Figure 1: Kernel convergence of the complement of slits.

Definition 2.4. Fix $T > 0$ and let $\mathcal{P}(\mathbb{R})$ be the space of probability measures on \mathbb{R} endowed with the topology of weak convergence (which is a metric space due to the well-known Lévy-Prokhorov metric). We denote by $\mathcal{M}(T) = C([0, T], \mathcal{P}(\mathbb{R}))$ the space of all continuous measure-valued processes on $[0, T]$ endowed with the topology of uniform convergence.

For every $N \in \mathbb{N}$, $\alpha_{N,t}$ can be regarded as a random element from $\mathcal{M}(T)$.

2.4 Tightness

We call a sequence $\{\mu_N\}_{N \in \mathbb{N}}$ of random elements from $C([0, T], \mathbb{R})$ (or $\mathcal{M}(T)$) *tight* if there exists a subsequence which converges in distribution. By Prohorov's Theorem ([Bil99, Section 5]), this coincides with the usual definition of tightness.

We are now going to list certain conditions that guarantee tightness of the sequence $\{\alpha_{N,t}\}_{N \in \mathbb{N}}$ defined in (2.9).

First of all, we make the following assumption:

$$\text{there exists } C > 0 \text{ such that for every } N \in \mathbb{N} \text{ it holds } \max_{k \in \{1, \dots, N\}} \lambda_{N,k} \leq \frac{C}{N}. \quad (\text{a})$$

Now, we introduce the “empirical distribution”

$$\mu_{N,t} = \sum_{k=1}^N \frac{1}{N} \delta_{V_{N,k}(t)}$$

and we let $L_N : [0, 1] \rightarrow [0, 1]$ be defined as $L_N(k/N) = \sum_{j=1}^k \lambda_{N,j}$ for $k = 0, \dots, N$. Next, we extend L_N to the entire unit interval $[0, 1]$ by linear interpolation. Then the family $\{L_N\}_{N \in \mathbb{N}}$ is uniformly bounded by 1 and equicontinuous by (a). The Ascoli–Arzelà Theorem implies that it is precompact. We will hence assume that the limit exists:

$$L_N(x) \rightarrow L(x) \text{ uniformly on } [0, 1] \text{ as } N \rightarrow \infty. \quad (\text{b})$$

Notice that if $F_{N,t}(x) = \alpha_{N,t}(-\infty, x]$ and $G_{N,t}(x) = \mu_{N,t}(-\infty, x]$ are the cumulative distribution functions, we have that

$$F_{N,t}(x) = L_N(G_{N,t}(x)). \quad (2.12)$$

Finally, the last assumption is rather a technical condition. Namely, we assume that $\mu_{N,0}$ converges weakly to a probability measure μ in such a way that there exists a C^2 -function $\varphi : \mathbb{R} \rightarrow [1, \infty)$, with φ', φ'' bounded and $\varphi(x) \rightarrow \infty$ for $x \rightarrow \pm\infty$ such that

$$\sup_{N \in \mathbb{N}} \int_{\mathbb{R}} \varphi(x) d\mu_{N,0}(x) < +\infty. \quad (\text{c})$$

Let $C_b^2(\mathbb{R}, \mathbb{C})$ be the space of all twice continuously differentiable functions $f : \mathbb{R} \rightarrow \mathbb{C}$ such that f, f', f'' are bounded.

Theorem 2.5. *Let $T > 0$. Then, under the assumptions (a), (b) and (c), the sequences $\{\mu_{N,t}\}_N$ and $\{\alpha_{N,t}\}_N$ are tight with respect to $\mathcal{M}(T)$.*

Moreover, if $\mu_{N_k,t}$ is a converging subsequence of $\{\mu_{N,t}\}_N$ with limit μ_t , then

1. $\alpha_{N_k,t}$ converges to the process α_t , and for every $t \in [0, T]$ the cumulative distribution function F_t of α_t is given by

$$F_t(x) = L \circ G_t(x) \quad (2.13)$$

where G_t is the cumulative distribution function of μ_t ;

2. μ_t satisfies the (distributional) differential equation

$$\begin{cases} \frac{d}{dt} \left(\int_{\mathbb{R}} f(x) d\mu_t(x) \right) = 2 \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} d\mu_t(x) d\alpha_t(y) \\ \mu_0 = \mu \end{cases} \quad (2.14)$$

for all $f \in C_b^2(\mathbb{R}, \mathbb{C})$.

Remark 2.6. *The conditions (b) and (c) are natural in the sense that we should assume convergence of the initial conditions $x_{N,k}$ and the coefficients $\lambda_{N,k}$, which are encoded in the functions L_N . If some $\lambda_{N,k}$ do not converge to 0 as $N \rightarrow \infty$, then some part of the measure $\alpha_{N,t}$ may escape to infinity as $N \rightarrow \infty$, see Example 2.16.*

Proof. To begin with, we notice that proving tightness of $\{\mu_{N,t}\}_N$ can be reduced to proving tightness of stochastic *real-valued* processes (see [RS93, Section 3] and also [Gär88, Section 1.3]). Thus, the sequence $\{\mu_{N,t}\}_N$ is tight if

$$\left\{ \int_{\mathbb{R}} \varphi(x) d\mu_{N,t}(x) \right\}_N \quad \text{and} \quad \left\{ \int_{\mathbb{R}} f(x) d\mu_{N,t}(x) \right\}_N$$

are tight sequences (with respect to the space $C([0, T], \mathbb{R})$ with uniform convergence) for all $f \in C_b^2(\mathbb{R}, \mathbb{C})$.

Now, let $f \in C_b^2(\mathbb{R}, \mathbb{C})$; Itô’s formula gives

$$\begin{aligned}
d\left(\int_{\mathbb{R}} f(x) d\mu_{N,t}(x)\right) &= d\left(\sum_{k=1}^N \frac{1}{N} f(V_{N,k}(t))\right) \\
&= \sum_{k=1}^N \frac{1}{N} \left(f'(V_{N,k}(t)) \sum_{j \neq k} \frac{2(\lambda_{N,k} + \lambda_{N,j})}{V_{N,k}(t) - V_{N,j}(t)} dt + \frac{\kappa \lambda_{N,k}}{2} f''(V_{N,k}(t)) dt + f'(V_{N,k}(t)) \sqrt{\kappa \lambda_{N,k}} dB_{N,k}(t) \right) \\
&= \sum_{k=1}^N \frac{1}{N} f'(V_{N,k}(t)) \sum_{j \neq k} \frac{2\lambda_{N,j}}{V_{N,k}(t) - V_{N,j}(t)} dt + \sum_{k=1}^N \frac{\lambda_{N,k}}{N} f'(V_{N,k}(t)) \sum_{j \neq k} \frac{2}{V_{N,k}(t) - V_{N,j}(t)} dt + \dots \\
&= \iint_{x \neq y} \frac{2f'(x)}{x-y} d\mu_{N,t}(x) d\alpha_{N,t}(y) dt + \iint_{x \neq y} \frac{2f'(x)}{x-y} d\alpha_{N,t}(x) d\mu_{N,t}(y) dt + \dots \\
&= 2 \iint_{x \neq y} \frac{f'(x) - f'(y)}{x-y} d\mu_{N,t}(x) d\alpha_{N,t}(y) dt + \dots \\
&= \underbrace{2 \iint_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x-y} d\mu_{N,t}(x) d\alpha_{N,t}(y) dt}_{=: A_N(t)} + \underbrace{2 \sum_{k=1}^N \frac{\lambda_{N,k}}{N} f''(V_{N,k}(t)) dt}_{=: B_N(t)} \\
&\quad + \underbrace{\frac{\kappa}{2} \sum_{k=1}^N \frac{\lambda_{N,k}}{N} f''(V_{N,k}(t)) dt}_{=: C_N(t)} + \underbrace{\sum_{k=1}^N \frac{1}{N} f'(V_{N,k}(t)) \sqrt{\kappa \lambda_{N,k}} dB_{N,k}(t)}_{=: D_N(t)}.
\end{aligned}$$

As f' and f'' are bounded and $\lambda_{N,k} \leq C/N$, it is easy to see that $A_N(t)$ is uniformly bounded and that $B_N(t), C_N(t), D_N(t)$ all converge to 0 as $N \rightarrow \infty$. By the “stochastic Ascoli–Arzelà Theorem” ([Bil99, Thm. 7.3]), we conclude that $\{\int_{\mathbb{R}} f(x) d\mu_{N,t}(x)\}_{N \in \mathbb{N}}$ is tight. Plus, in view of the boundedness of both φ' and φ'' , thanks to assumption (c), the same reasoning also implies tightness of the sequence $\{\int_{\mathbb{R}} \varphi(x) d\mu_{N,t}(x)\}_{N \in \mathbb{N}}$. Hence, $\mu_{N,t}$ is tight and each limit process satisfies equation (2.14).

Finally, it follows from (2.12) and assumption (b) that the subsequence $\alpha_{N_k,t}$ converges provided the convergence of $\mu_{N_k,t}$. In particular, it follows that relation (2.13) holds for the limit processes. \square

Now we can easily show that if $\mu_{N_k,t}$ is a converging subsequence, then $g_{N_k,t}$ converges as well.

First, let \mathcal{C} be the set of all $M(z) = \int_{\mathbb{R}} \frac{2}{z-u} d\beta(u)$, where β is a probability measure. So M is 2 times the Cauchy transform (or Stieltjes transform) of β . The measure β can be recovered from M by the Stieltjes–Perron inversion formula (see [Sch12, Theorem F.2]). Denote its distribution function by $F(x)$. Then $L \circ F(x)$ is also a distribution function, which corresponds to a measure $\hat{\beta}$. In this way, we obtain a map $\mathcal{L} : \mathcal{C} \rightarrow \mathcal{C}$ defined as

$$\int_{\mathbb{R}} \frac{2}{z-u} d\beta(u) \mapsto \int_{\mathbb{R}} \frac{2}{z-u} d\hat{\beta}(u).$$

The limit of the Loewner equation can now be described as follows.

Corollary 2.7. *Let $\mu_{N_k,t}$ be a converging subsequence with limit μ_t . Then $g_{N_k,t}$ converges in distribution with respect to locally uniform convergence to g_t , the solution of the Loewner equation*

$$\frac{d}{dt} g_t = (\mathcal{L} \circ M_t)(g_t), \quad (2.15)$$

where $M_t = \int_{\mathbb{R}} \frac{2}{z-u} d\mu_t(u)$ solves the (abstract) differential equation

$$\begin{cases} \frac{\partial}{\partial t} M_t = -\frac{\partial}{\partial z} M_t \cdot (\mathcal{L} \circ M_t) - M_t \cdot \frac{\partial}{\partial z} (\mathcal{L} \circ M_t), \\ M_0(z) = \int_{\mathbb{R}} \frac{2}{z-u} d\mu(u). \end{cases} \quad (2.16)$$

Remark 2.8. *The convergence of $\alpha_{N,t}$ and $g_{N,t}$ would follow immediately if we knew that equation (2.16) (or, equivalently, (2.14)) had a unique solution. If $\lambda_{N,k} = \frac{1}{N}$, then (2.16) is a usual PDE and uniqueness can be shown easily (see Section 2.5).*

In order to prove the above corollary, we will need the following control-theoretic result.

Theorem 2.9. *Fix some $t > 0$. Let λ be the Lebesgue measure on $[0, t]$ and let $\mathcal{N}(t)$ be the space of all finite measures on $\mathbb{R} \times [0, t]$ endowed with the topology of weak convergence. Let $\{\beta_{N,s}\}_{N \in \mathbb{N}}$ be a sequence of processes from $\mathcal{M}(t)$ and assume $\beta_{N,s} \times \lambda \in \mathcal{N}(t)$ converges weakly to $\beta_s \times \lambda \in \mathcal{N}(t)$ as $N \rightarrow \infty$. Denote with $h_{N,s}$, $s \in [0, t]$, the solution to the Loewner equation*

$$\frac{d}{ds} h_{N,s}(z) = \int_{\mathbb{R}} \frac{2}{h_{N,s}(z) - u} d\beta_{N,s}(u), \quad h_{N,0}(z) = z.$$

Then $h_{N,t}$ converges locally uniformly to h_t , where h_s , $s \in [0, t]$, is the solution to

$$\frac{d}{ds} h_s(z) = \int_{\mathbb{R}} \frac{2}{h_s(z) - u} d\beta_s(u), \quad h_0(z) = z.$$

A proof of the above theorem can be found in [JVST12, Proposition 1] or [MS13, Theorem 1.1]. Notice that even though both results consider the radial Loewner equation, the proofs can be easily adapted to the chordal case.

Proof of Corollary 2.7. For $z \in \mathbb{H}$, let $f(x) = \frac{2}{z-x}$. Then $f \in C_b^2(\mathbb{R}, \mathbb{C})$. Define now $M_t(z) = \int_{\mathbb{R}} f(x) d\mu_t(x)$; then $(\mathcal{L} \circ M_t)(z) = \int_{\mathbb{R}} f(x) d\alpha_t(x)$, where α_t is the limit of $\alpha_{N_k,t}$, and Theorem 2.5 implies

$$\begin{aligned} \frac{\partial}{\partial t} M_t(z) &= 4 \int_{\mathbb{R}^2} \frac{\frac{1}{(z-x)^2} - \frac{1}{(z-y)^2}}{x-y} d\mu_t(x) d\alpha_t(y) = 4 \int_{\mathbb{R}^2} \frac{2z-x-y}{(z-x)^2(z-y)^2} d\mu_t(x) d\alpha_t(y) \\ &= \int_{\mathbb{R}^2} \frac{2}{(z-x)^2} \frac{2}{(z-y)} + \frac{2}{(z-x)} \frac{2}{(z-y)^2} d\mu_t(x) d\alpha_t(y) \\ &= -\frac{\partial}{\partial z} M_t \cdot (\mathcal{L} \circ M_t) - M_t \cdot \frac{\partial}{\partial z} (\mathcal{L} \circ M_t). \end{aligned}$$

Furthermore, let g_t be the solution to

$$\frac{d}{dt} g_t = (\mathcal{L} \circ M_t)(g_t), \quad g_0(z) = z.$$

Fix some $t > 0$. The canonical mapping $\mathcal{M}(t) \ni \alpha_s \mapsto \alpha_s \times \lambda \in \mathcal{N}(t)$ is continuous. It follows from the Continuous Mapping Theorem (see [Bil99], p. 20) that $\alpha_{N_k,s} \times \lambda$ converges in distribution with respect to weak convergence to $\alpha_s \times \lambda$.

Hence, Theorem 2.9 and again the Continuous Mapping Theorem imply that $g_{N_k,t}$, which is the solution to (2.10), converges in distribution to g_t with respect to locally uniform convergence. \square

2.5 The simultaneous case

In the case $\lambda_{N,k} = \frac{1}{N}$ for all k , which we call the *simultaneous* case, equation (2.7) becomes

$$dV_{N,k} = \sum_{j \neq k} \frac{4/N}{V_{N,k} - V_{N,j}} dt + \sqrt{\kappa/N} dB_{N,k}, \quad (2.17)$$

a process that is quite similar to a Dyson Brownian motion.

Note that in such a case $\mu_{N,t} = \alpha_{N,t}$ and \mathcal{L} is the identity map. If α_t is the limit of a converging subsequence of $\{\alpha_{N,t}\}_N$ and $M_t(z) = \int_{\mathbb{R}} \frac{2}{z-u} d\alpha_t(u)$, then M_t satisfies the complex Burgers equation

$$\begin{cases} \frac{\partial}{\partial t} M_t = -2M_t \cdot \frac{\partial}{\partial z} M_t(z) \\ M_0(z) = \int_{\mathbb{R}} \frac{2}{z-u} d\alpha_0(u) \end{cases}, \quad (2.18)$$

and the limit of $g_{N_k,t}$ satisfies

$$\frac{d}{dt} g_t = M_t(g_t), \quad g_0(z) = z. \quad (2.19)$$

If we put $f_t = g_t^{-1}$, we obtain the Loewner PDE mentioned in Section 1:

$$\frac{\partial f_t(z)}{\partial t} = -\frac{\partial f_t(z)}{\partial z} \cdot M_t(z).$$

Theorem 2.10. *Under the assumptions of Theorem 2.5 with $\lambda_{N,k} = \frac{1}{N}$, the sequences $\alpha_{N,t}$ and $g_{N,t}$ converge in distribution as $N \rightarrow \infty$.*

As already mentioned, this follows as soon as we know that equation (2.18) has a unique solution, which is shown, e.g., in [RS93, Section 4] or [CL97, Section 5]. We give here another short proof.

Proof. Let M_t be a solution of (2.18). As M_t has no zeros in \mathbb{H} we can consider $F_t := 1/M_t$ which satisfies $\frac{\partial}{\partial t} F_t = -2F_t^{-1} \cdot \frac{\partial}{\partial z} F_t$. Next we use the fact that every F_t is univalent in a region $\Gamma_{\alpha(t), \beta(t)}$, where

$$\Gamma_{\alpha, \beta} := \{z \in \mathbb{H} \mid \operatorname{Im}(z) > \beta, \operatorname{Im}(z) > \alpha |\operatorname{Re}(z)|\}, \alpha, \beta > 0,$$

see [BV93, Proposition 5.4]. For $t \in [0, T]$ we find α_0 and β_0 such that F_t is univalent in $\Gamma_{\alpha_0, \beta_0}$ for all $t \in [0, T]$.

Thus we can define $V_t(z)$ as $F_t^{-1}(z)$ for $z \in \Gamma_{\alpha_0, \beta_0}$ and a simple calculation gives

$$\frac{\partial}{\partial t} V_t(z) = \frac{2}{z}, \quad V_0(z) = (1/M_0)^{-1}(z).$$

Obviously, V_t , and hence also M_t , is uniquely determined. \square

Remark 2.11. *Transforms like $\mu_t \mapsto V_t(z)$ appear in free probability theory, which was introduced by D. Voiculescu in the 1980's (in [AEPA09, p. 3059], $V_t(z) - z$ is called Voiculescu transform). We notice that Wigner's semicircle law appears here as follows: for $\alpha_0 = \delta_0$, the solution of (2.18) is given by $M_t(z) = \frac{4}{z + \sqrt{z^2 - 16t}}$, which is 2 times the Cauchy transform of the centred semicircle law with variance $4t$.*

For relations between the chordal (and radial) Loewner equation to non-commutative probability theory, we refer to [Bau04, Sch16].

Remark 2.12. *In [dMS16], the authors prove some geometric properties of the solution g_t of (2.19), under the assumption that the support of α_0 is bounded. We mention one property of this case, which will be needed later on.*

The measures α_t “grow” continuously in the following sense: $\operatorname{supp} \alpha_s \subset \operatorname{supp} \alpha_t$ for all $s \leq t$ and for each $x \in \mathbb{R} \setminus \operatorname{supp} \alpha_s$ there exists $T > s$ such that $x \notin \operatorname{supp} \alpha_t$ for all $t \leq T$. This is actually a consequence of the theory of the real Burgers equation (see [dMS16, Section 3.4]).

Remark 2.13. *Let M_t be a solution of (2.18) and $c > 0$. Define $G_t(z) := c \cdot M_{c^2 t}(c \cdot z)$. Then G_t also satisfies (2.18) with initial value $G_0(z) = c \cdot M_0(c \cdot z)$. Fix some $T > 0$. As $G_0(z) \rightarrow \frac{2}{z}$ when $c \rightarrow \infty$, we obtain together with Remark 2.11 the long time behaviour*

$$\lim_{c \rightarrow \infty} c \cdot M_{c^2 T}(c \cdot z) = \frac{4}{z + \sqrt{z^2 - 16T}} \quad \text{or} \quad M_t(z) \sim \frac{4}{z + \sqrt{z^2 - 16t}} \text{ as } t \rightarrow \infty.$$

2.6 Examples

In the following we consider three examples. In all three cases we assume that $\kappa = 0$, i.e. we look at the deterministic case to make the differential equations somewhat simpler.

The proof of Theorem 2.5 shows that the sequence $\frac{d}{dt} \left(\int_{\mathbb{R}} f(x) d\mu_{N,t}(x) \right)$, as a sequence of functions on $[0, T]$, is uniformly bounded. In general, this is not true for $\alpha_{N,t}$.

Example 2.14. *Let $S_N = 1 + \frac{N+1}{2N}$. We choose*

$$x_{N,k} = \frac{k}{N^2} \quad \text{and} \quad \lambda_{N,k} = \frac{1}{S_N} \left(1 + \frac{k}{N} \right) \cdot \frac{1}{N}.$$

Obviously,

$$\alpha_{N,0}, \mu_{N,0} \xrightarrow{\mathbf{w}} \delta_0,$$

and $\lambda_{N,k} \leq C/N$ for some $C > 0$ as $\lambda_{N,k} \leq \lambda_{N,N} \sim \frac{2}{3N}$ as $N \rightarrow \infty$. Furthermore, as $x_{N,k} \in [0, 1]$ for all k, N , it is easy to see that condition (c) is satisfied.

Finally, $L_N(k/N)$ is given by $L_N(k/N) = \sum_{j=1}^k \lambda_{N,j} = \frac{1}{S_N} \left(\frac{k}{N} + \frac{k}{N} \cdot \frac{k+1}{2N} \right)$, which shows that L_N converges uniformly to $L(x) = \frac{2}{3} \left(x + \frac{x^2}{2} \right)$. Consequently, all the assumptions of Theorem 2.5 are satisfied.

Proposition 2.15. *Under the assumptions of Example 2.14, there exists $f \in C_b^2(\mathbb{R}, \mathbb{C})$ such that $\frac{d}{dt} \left(\int_{\mathbb{R}} f(x) d\alpha_{N,t}(x) \right) |_{t=0}$ is unbounded.*

Proof. Note that

$$\frac{\lambda_{N,k} - \lambda_{N,j}}{x_{N,k} - x_{N,j}} = 1/S_N \frac{k/N^2 - j/N^2}{k/N^2 - j/N^2} = 1/S_N. \quad (*)$$

Let $f \in C_b^2(\mathbb{R}, \mathbb{C})$. Then we obtain

$$\begin{aligned} \frac{d}{dt} \left(\int_{\mathbb{R}} f(x) d\alpha_{N,t}(x) \right) &= \frac{d}{dt} \left(\sum_{k=1}^N \lambda_{N,k} f(V_{N,k}(t)) \right) = \sum_{k=1}^N \lambda_{N,k} f'(V_{N,k}(t)) \sum_{j \neq k} \frac{2(\lambda_{N,k} + \lambda_{N,j})}{V_{N,k}(t) - V_{N,j}(t)} \\ &= \sum_{k=1}^N \lambda_{N,k} f'(V_{N,k}(t)) \sum_{j \neq k} \frac{2\lambda_{N,j}}{V_{N,k}(t) - V_{N,j}(t)} + \sum_{k=1}^N \lambda_{N,k}^2 f'(V_{N,k}(t)) \sum_{j \neq k} \frac{2}{V_{N,k}(t) - V_{N,j}(t)} \\ &= \iint_{x \neq y} \frac{2f'(x)}{x - y} d\alpha_{N,t}(x) d\alpha_{N,t}(y) + 2 \sum_{j \neq k} \frac{\lambda_{N,k}^2 f'(V_{N,k}(t))}{V_{N,k}(t) - V_{N,j}(t)} \\ &= \iint_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} d\alpha_{N,t}(x) d\alpha_{N,t}(y) + \sum_{k=1}^N \lambda_{N,k}^2 f''(V_{N,k}(t)) \\ &\quad + \underbrace{\sum_{j \neq k} \frac{\lambda_{N,k}^2 f'(V_{N,k}(t)) - \lambda_{N,j}^2 f'(V_{N,j}(t))}{V_{N,k}(t) - V_{N,j}(t)}}_{:= T_N(t)}. \end{aligned}$$

Now assume that $f'(x) = 1$ for all $x \in [0, 1]$. It is easy to see that the first two terms are uniformly bounded. However,

$$\begin{aligned} T_N(0) &= \sum_{j \neq k} \frac{\lambda_{N,k}^2 - \lambda_{N,j}^2}{x_{N,k} - x_{N,j}} \stackrel{(*)}{=} \sum_{j \neq k} (\lambda_{N,k} + \lambda_{N,j})/S_N \geq \sum_{j \neq k} \lambda_{N,1}/S_N \\ &= \frac{(N^2 - N)(\frac{1}{N} + \frac{1}{N^2})}{S_N^2} \rightarrow \infty \quad \text{as } N \rightarrow \infty. \end{aligned}$$

□

Next we have a look at two examples where condition (a) is not satisfied, as for every N there is one coefficient $\lambda_{N,k} = \frac{1}{2}$.

Example 2.16. For $N \geq 2$, let

$$x_{N,k} = \frac{k}{N} \text{ and } \lambda_{N,k} = \frac{1}{2(N-1)} \text{ for all } k \neq N, \text{ and } x_{N,N} = 2, \lambda_{N,N} = \frac{1}{2}.$$

Proposition 2.17. Let $T > 0$. Under the assumptions of Example 2.16, the sequence $\{\alpha_{N,t}\}_N$ is not tight with respect to the topology of $\mathcal{M}(T)$.

Proof. We show that $V_{N,N}(t) \rightarrow +\infty$ as $N \rightarrow \infty$ for every $t > 0$. As $V_{N,N}$ carries the mass $1/2$, this proves that $\{\alpha_{N,t}\}_N$ is not tight.

First, we need an upper bound for $V_{N,N-1}$. For $k \in 1, \dots, N-1$ we have

$$dV_{N,k}(t) \leq \sum_{j \neq k, N} \frac{\frac{2}{N-1}}{V_{N,k}(t) - V_{N,j}(t)} dt.$$

Let $W_{N,1}, \dots, W_{N,N-1}$ be the of solution the system

$$dW_{N,k}(t) = \sum_{j \neq k, N} \frac{\frac{2}{N-1}}{W_{N,k}(t) - W_{N,j}(t)} dt, \quad W_{N,k}(0) = x_{N,k}.$$

As the function

$$(x_1, \dots, x_{N-1}) \mapsto \left(\sum_{j \neq 1, N} \frac{\frac{2}{N-1}}{x_1 - x_j}, \dots, \sum_{j \neq N-1, N} \frac{\frac{2}{N-1}}{x_{N-1} - x_j} \right)$$

is quasimonotone, it follows that $V_{N,k}(t) \leq W_{N,k}(t)$ for all $t \geq 0$ (Theorem 4.2 in [LL80]). Note that $W_{N,1}, \dots, W_{N,N-1}$ is a simultaneous multiple SLE process for $N-1$ curves, each growing with “speed” $\frac{1}{2(N-1)}$. From Remark 2.12 we conclude that there exists $T_0 > 0$ and a bound $B_1 \in (1, 2)$ such that $W_{N,N-1}(t) \leq B_1$ for all $t \in [0, T_0]$ and $N \geq 2$. Hence, $V_{N,k}(t) \leq B_1 < 2$ for all $t \in [0, T_0]$. This upper bound now gives us also a lower bound as follows:
As $\frac{d}{dt} V_{N,N}(t) \geq 0$ and $V_{N,N}(0) = 2$ we have $V_{N,N}(t) \geq 2$ for all $t \geq 0$. Thus, for $k \in \{1, \dots, N-1\}$ we have

$$\begin{aligned} dV_{N,k}(t) &= \sum_{j \neq k, N} \frac{\frac{2}{N-1}}{V_{N,k}(t) - V_{N,j}(t)} dt + \frac{2(1/2 + \frac{1}{2(N-1)})}{V_{N,k}(t) - V_{N,N}(t)} dt \\ &\geq \sum_{j \neq k, N} \frac{\frac{2}{N-1}}{V_{N,k}(t) - V_{N,j}(t)} dt + \frac{2}{B_1 - 2} dt. \end{aligned}$$

Let $Y_{N,1}, \dots, Y_{N,N-1}$ be the of solution the system

$$dY_{N,k}(t) = \sum_{j \neq k, N} \frac{\frac{2}{N-1}}{Y_{N,k}(t) - Y_{N,j}(t)} dt + \frac{2}{B_1 - 2} dt, \quad Y_{N,k}(0) = x_{N,k}.$$

From [CL97], Theorem 5.1, it follows that the sequence $w_{N,t} = \sum_{k=1}^{N-1} \frac{1}{N-1} \delta_{Y_{N,k}(t)}$ of measure-valued processes converges as $N \rightarrow \infty$. This does not imply that $Y_{N,1}(t)$ is bounded from below, but we can conclude that, for example, $Y_{N, \lfloor N/2 \rfloor}(t)$ is bounded from below, i.e. there exists $B_2 < 1$ such that $Y_{N, \lfloor N/2 \rfloor}(t) \geq B_2$ for all $t \in [0, T_0]$.
Now we look at $V_{N,N}$, which satisfies

$$dV_{N,N} = \sum_{j \neq N} \frac{2(\frac{1}{2} + \frac{1}{2(N-1)})}{V_{N,N}(t) - V_{N,j}(t)} dt \geq \sum_{j \leq \lfloor N/2 \rfloor} \frac{1}{V_{N,N}(t) - B_2} dt = \frac{\lfloor N/2 \rfloor}{V_{N,N}(t) - B_2} dt$$

for $t \in [0, T_0]$, which implies

$$V_{N,N}(t) \geq B_2 + \sqrt{4 - 4B_2 + B_2^2 - 2t + 2\lfloor N/2 \rfloor t}.$$

Hence, $V_{N,N}(t) \rightarrow \infty$ for every $t \in (0, T_0]$ as $N \rightarrow \infty$. As $t \mapsto V_{N,N}(t)$ is increasing, we conclude that $V_{N,N}(t) \rightarrow \infty$ for every $t > 0$. \square

Even though $\{\alpha_{N,t}\}_N$ is not tight in this example, it is easy to see that $g_{N,t}$ converges as $N \rightarrow \infty$. If we decompose $\alpha_{N,t} = \beta_{N,t} + \frac{1}{2} \delta_{V_{N,N}(t)}$, then it can easily be shown that $\beta_{N,t}$ converges to a process β_t and that $P_t(z) = \int_{\mathbb{R}} \frac{2}{z-u} d\beta_t(u)$ satisfies a Burgers equation.

Example 2.18. Assume that $N = 2K + 1$, $K \in \mathbb{N}$, and let

$$x_{N,k} \in [-2, -1] \text{ and } x_{N,2K+2-k} = -x_{N,k} \in [1, 2] \text{ for all } k \leq K.$$

Assume that $x_{N,K+1} = 0$. The coefficients $\lambda_{N,k}$ are chosen as

$$\lambda_{N,K+1} = 1/2, \quad \lambda_{N,k} = \frac{1}{4K}, k \neq K+1.$$

As $N \rightarrow \infty$, the sequence L_N converges pointwise, but not uniformly, to $L(x) = 1/2x$, $x \in [0, 1/2)$, $L(x) = 1/2x + 1/2$, $x \in [1/2, 1]$.

Proposition 2.19. Under the assumptions of example 2.18, there exists $T_0 > 0$ such that the sequence $\{\alpha_{N,t}\}_N$ is tight with respect to the topology of $\mathcal{M}(T_0)$.

Proof. By symmetry, we have $V_{N,K+1}(t) = 0$ for every $K \in \mathbb{N}$ and $t \geq 0$ and we can decompose the measure $\alpha_{N,t}$ as $\alpha_{N,t} = \beta_{N,t} + \frac{1}{2} \delta_0 + \gamma_{N,t}$, where the support of $\beta_{N,t}$ is contained in $(-\infty, 0)$ and $\gamma_{N,t}(A) = \beta_{N,t}(-A)$ for every Borel set A .

Just as in the proof of Proposition 2.17, we obtain that there exist $T_0 > 0$ and $B \in (-1, 0)$ such that

$$V_{N,K}(t) \leq B \text{ for all } K \in \mathbb{N} \text{ and } t \in [0, T_0]. \quad (2.20)$$

Now let $f \in C_b^2(\mathbb{R}, \mathbb{C})$. Then we have

$$\begin{aligned}
\frac{d}{dt} \left(\int_{\mathbb{R}} f(x) d\beta_{N,t}(x) \right) &= \frac{d}{dt} \left(\sum_{k=1}^K \frac{1}{4K} f(V_{N,k}(t)) \right) \\
&= \sum_{k=1}^K \frac{f'(V_{N,k}(t))}{4K} \left(\sum_{\substack{j \leq K \\ j \neq k}} \frac{2(\lambda_{N,k} + \lambda_{N,j})}{V_{N,k}(t) - V_{N,j}(t)} + \sum_{\substack{j \geq K+2 \\ j \neq k}} \frac{2(\lambda_{N,k} + \lambda_{N,j})}{V_{N,k}(t) - V_{N,j}(t)} + \frac{2(\lambda_{N,k} + \lambda_{N,K+1})}{V_{N,k}(t) - V_{N,K+1}(t)} \right) \\
&= \sum_{k=1}^K \frac{f'(V_{N,k}(t))}{4K} \left(\sum_{\substack{j \leq K \\ j \neq k}} \frac{\frac{1}{K}}{V_{N,k}(t) - V_{N,j}(t)} + \sum_{\substack{j \leq K \\ j \neq k}} \frac{\frac{1}{K}}{V_{N,k}(t) + V_{N,j}(t)} + \frac{\frac{1}{2K} + 1}{V_{N,k}(t)} \right) \\
&= 4 \iint_{x \neq y} \frac{f'(x)}{x-y} + \frac{f'(x)}{x+y} d\beta_{N,t}(x) d\beta_{N,t}(y) + \sum_{k=1}^K \frac{f'(V_{N,k}(t))(\frac{1}{2K} + 1)}{4K V_{N,k}(t)} \\
&= 2 \iint_{x \neq y} \frac{f'(x) - f'(y)}{x-y} + \frac{f'(x) + f'(y)}{x+y} d\beta_{N,t}(x) d\beta_{N,t}(y) + \sum_{k=1}^K \frac{f'(V_{N,k}(t))(\frac{1}{2K} + 1)}{4K V_{N,k}(t)}.
\end{aligned}$$

The term $\frac{f'(x) - f'(y)}{x-y}$ is bounded as $f \in C_b^2(\mathbb{R}, \mathbb{C})$, and the integral over $\frac{f'(x) + f'(y)}{x+y}$ is bounded for $t \in [0, T_0]$ because of (2.20). The sum is also bounded for all $t \in [0, T_0]$ because of (2.20). We conclude that $\{\alpha_{N,t}\}_N$ and thus $\{\alpha_{N,t}\}_N$ is tight w.r.t. $\mathcal{M}(T_0)$. \square

It seems that the last two example behave in the same way when $\kappa > 0$. Figures 2 and 3 show simulations of the driving functions $V_{N,1}, \dots, V_{N,N}$ for these two cases on the time interval $[0, 1]$ for $N = 51$ and $\kappa = 1$. The driving function with mass $\frac{1}{2}$ is coloured red.

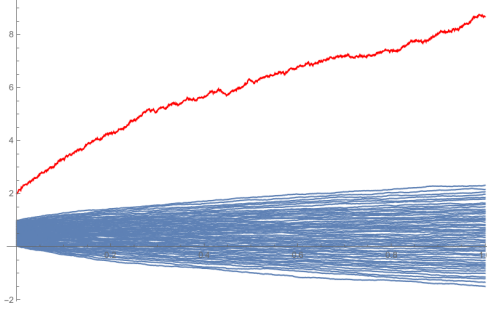


Figure 2: Mass $\frac{1}{2}$ in $x_{N,N}$.

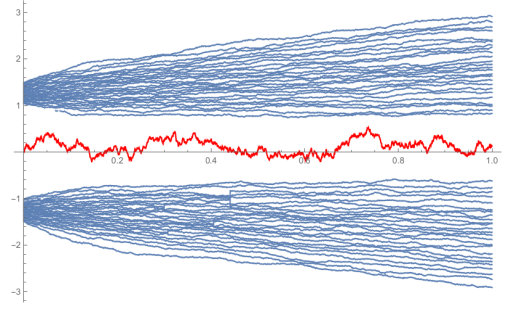


Figure 3: Mass $\frac{1}{2}$ in $x_{N,(N-1)/2}$.

2.7 Problems and Remarks

1. As already mentioned in Remark 2.8, the convergence of $g_{N,t}$ from Corollary 2.7 follows as soon as we know that equation (2.16) has only one solution.
2. Example 2.14 suggests that the process α_t might not in general be differentiable (in the distributional sense) at $t = 0$.

Question: Is it always differentiable for $t > 0$?

Also, we notice that in [BBCL99] it is shown that, for a special case, α_t has a density with respect to the Lebesgue measure for $t > 0$.

Question: Is this always true for α_t under the assumptions made in Theorem 2.5?

3. Fix a parameter $\kappa \in (0, 4]$. For each $N \in \mathbb{N}$, we consider $2N$ boundary points $0 < p_{N,1} < p_{N,2} < \dots < p_{N,2N} = 1$ for multiple SLE on \mathbb{H} . We set $\mathbf{p}_N := (p_{N,1}, \dots, p_{N,2N})$. Recall that $S(\mathbf{p}_N)$ is the set of all C_N configurations for these points, endowed with the probabilities given by formula (2.5).

Now we can ask for the limit of $S(\mathbf{p}_N)$ as $N \rightarrow \infty$ by using an idea from combinatorics, to encode configurations into Dyck paths.

An N -Dyck path is a continuous function $d : [0, 2N] \rightarrow [0, \infty)$ defined as follows:

- $d(0) = 0$ and $d(2N) = 0$,
- $d(k) - d(k+1) \in \{-1, +1\}$ for all $k \in \{0, \dots, 2N-1\}$,
- for all other points $x \in [0, 2N] \setminus \{0, 1, \dots, 2N\}$, $d(x)$ is defined by linear interpolation.

The set of all N -Dyck paths corresponds to the set $S(\mathbf{p}_N)$ in the following way.

An N -Dyck path can be completely described by $2N$ numbers $L_1, \dots, L_{2N} \in \{-1, +1\}$ representing the slopes of the $2N$ line segments. These numbers are determined by a configuration for \mathbf{p}_N as follows (see the figures below for an example):

- (i) $L_k = +1$ and $L_{k+1} = -1$ if and only if p_k and p_{k+1} are connected by a simple curve;
- (ii) $L_k = L_{k+1} = +1$ if and only if the curve connecting p_{k+1} is “contained” in the curve connecting p_k ;
- (iii) $L_k = L_{k+1} = -1$ if and only if the curve connecting p_k is “contained” in the curve connecting p_{k+1} .

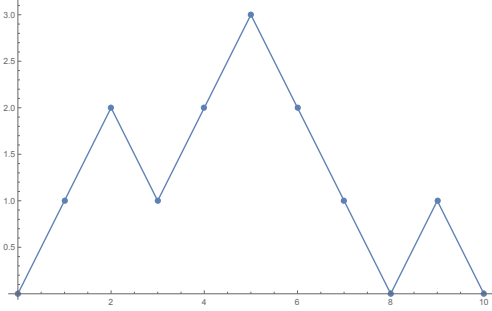


Figure 4: A Dyck path for $N = 5$.

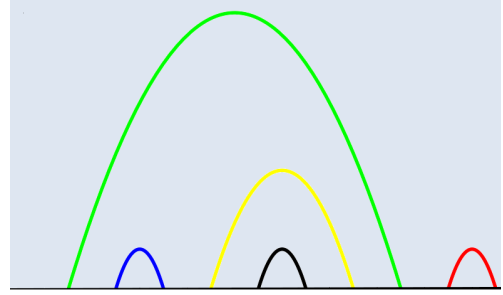


Figure 5: The configuration corresponding to Figure 1.

Define also $p_{N,0} := 0$ and fix some $\gamma \in (0, 1]$. Normalize now such a Dyck path d to define a normalized Dyck path as a continuous function $e_N : [0, 1] \rightarrow [0, \infty)$ with $e_N(p_{N,0}) = 0$ and

$$e_N(t) = e_N(p_{N,k}) + t \cdot \frac{d(k+1) - d(k)}{(p_{N,k+1} - p_{N,k})^\gamma} \quad (2.21)$$

for $t \in [p_{N,k}, p_{N,k+1}]$, $k = 0, \dots, 2N-1$. Then the set of all normalized Dyck paths is a subset of the space $C([0, 1], \mathbb{R})$ endowed with the topology of uniform convergence. It becomes a probability space by taking the corresponding probabilities from the set $S(\mathbf{p}_N)$. Let $E_N(t)$ be a random path from this set.

Question: Does $E_N(t)$ converge in distribution as $N \rightarrow \infty$?

Remark 2.20. Take $p_k = \frac{k}{2N}$ and $\gamma = \frac{1}{2}$. If all the probabilities are equally distributed, i.e. the probability for each normalized Dyck path is $\frac{1}{C_N}$, then the corresponding random path $E_N(t)$ converges in distribution to a Brownian excursion process of duration 1 (see [Ric09, Section 1.2] and [MM03]).

Furthermore, we note that the probabilities for configurations are also considered for $\kappa > 4$, e.g. in [KP15].

Question: What can be said about the limit of the probabilities for the set $S(\mathbf{p}_N)$ as $\kappa \rightarrow 0$?

4. The above questions can be extended to different settings like radial multiple SLE or multiple SLE in multiply connected domains (refer to [Law11]). For instance, in [Car03], the author describes the Loewner equation for radial SLE where N simple curves grow from the boundary of the unit disc \mathbb{D} within \mathbb{D} towards the interior point 0. The radial analogue of Theorem 2.10, i.e. the coefficients in the Loewner equation are $\frac{1}{N}$, can be obtained simply by using the main result of [CL01].

3 Trajectories of a certain quadratic differential

Finally, we take a look at a Loewner equation that describes the growth of N trajectories of a certain quadratic differential. By using the methods from the previous section, we obtain again an abstract differential equation for the limit case $N \rightarrow \infty$, which reduces to the Burgers equation in a special case.

Consider again N points $x_{N,1} < x_{N,2} < \dots < x_{N,N}$ on \mathbb{R} and the quadratic differential

$$Q_N(z) dz^2 = \prod_{k=1}^N (z - x_{N,k})^2 \prod_{j=1}^{M_N} (z - s_{N,j})^{\alpha_{N,j}} \prod_{j=1}^{M_N} (z - \overline{s_{N,j}})^{\alpha_{N,j}} dz^2,$$

where $M_N \in \mathbb{N} \cup \{0\}$, $s_{N,j} \in \mathbb{H}$ and $\alpha_{N,j} \in \mathbb{Z}$.

Then, \mathbb{R} is a trajectory of $Q(z)dz^2$ and, for every k , there is exactly one trajectory starting from $x_{N,k}$ and going into the upper half-plane. As Q has a zero of order 2 at $z = x_{N,k}$, this trajectory will form a 90° -angle with \mathbb{R} (see [Pom75, p. 213-215]).

Choose N coefficients $\lambda_{N,k} \in [0, 1]$ such that $\sum_{k=1}^N \lambda_{N,k} = 1$. The Loewner equation

$$g_{N,t}(z) = \sum_{k=1}^N \frac{2\lambda_{N,k}}{g_{N,t}(z) - V_{N,k}(t)}, \quad g_0(z) = z,$$

generates exactly these N curves provided that the driving functions $V_{N,k}(t)$ satisfy

$$\begin{cases} \frac{d}{dt} V_{N,k}(t) = \sum_{j \neq k} \frac{2\lambda_{N,j}}{V_{N,k}(t) - V_{N,j}(t)} + \sum_{j=1}^{M_N} \frac{\alpha_{N,j} \lambda_{N,k}}{V_{N,k}(t) - s_{N,j}(t)} + \sum_{j=1}^{M_N} \frac{\alpha_{N,j} \lambda_{N,k}}{V_{N,k}(t) - \overline{s_{N,j}(t)}} \\ V_{N,k}(0) = x_{N,k} \end{cases}$$

where $s_{N,j}(t) = g_{N,t}(s_{N,j})$.

Remark 3.1. This follows from [Tsa09, Theorem 5.1], where all the degrees μ_k^\pm are equal to 0 in our case, as the trajectories form a 90° -angle with the real axis, which is also a trajectory of $Q(z)dz^2$ (check p. 564 in [Tsa09]).

Next, define the probability measure $\mu_{N,t} = \sum_{k=1}^N \lambda_{N,k} \delta_{V_{N,k}(t)}$.

Remark 3.2. If $M_N = 0$ for all $N \in \mathbb{N}$ and $\lambda_{N,k} \leq C/N$ for all k, N and some $C > 0$, then, by the proof of Theorem 2.5, it is easy to see that the following holds:

If $\mu_{N,0} \rightarrow \mu$ as $N \rightarrow \infty$ such that (c) is satisfied, then the limit μ_t of $\mu_{N,t}$ exists, and the transform $M_t(z) = \int_{\mathbb{R}} \frac{2}{z-u} d\mu_t(u)$ satisfies the Burgers equation

$$\frac{\partial}{\partial t} M_t = -M_t \cdot \frac{\partial}{\partial z} M_t(z), \quad M_0(z) = \int_{\mathbb{R}} \frac{2}{z-u} d\mu(u).$$

Note that this is equation (2.18) with the 2 replaced by 1. The limit g_t of $g_{N,t}$ satisfies $\frac{\partial}{\partial t} g_t = M_t(g_t)$ and a simple calculation shows that $\frac{\partial}{\partial t} M_t(g_t(z)) = 0$, which implies that $t \mapsto g_t(z_0)$, for $z_0 \in \mathbb{H}$ fixed, describes a straight line, and that $M_t(g_t(z)) = M_0(z)$.

Assume now $\lambda_{N,k} = \frac{1}{N}$ for all k and N .

First, we introduce a second measure-valued process

$$\sigma_{N,t} = \sum_{j=1}^{M_N} \frac{\alpha_{N,j}}{N} \delta_{s_{N,j}(t)},$$

and we assume that there exists a compact set $K \subset \mathbb{H}$ such that

$$\text{supp } \sigma_{N,0} \subset K \quad \text{for all } N \in \mathbb{N}. \quad (\text{d})$$

Theorem 3.3. Let $\lambda_{N,k} = \frac{1}{N}$ for all k and N . Assume that $\mu_{N,0}$ converges weakly to the probability measure μ such that (c) holds. Furthermore, assume that (d) holds and that $\sigma_{N,0}$ converges weakly to a finite (signed) measure σ as $N \rightarrow \infty$. Then there exists $T > 0$ such that $\{\mu_{N,t}\}_{N \in \mathbb{N}}$ is tight as a sequence in $\mathcal{M}(T)$.

Moreover, let $\mu_{N_k,t}$ be a converging subsequence with limit μ_t . Then the following two statements hold:

(i) $\sigma_{N_k,t}$ converges to a process σ_t and

$$\begin{aligned} \frac{d}{dt} \left(\int_{\mathbb{R}} f(x) d\mu_t(x) \right) &= \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} d\mu_t(x) d\mu_t(y) + 2\text{Re} \left(\int_{\mathbb{H}} \int_{\mathbb{R}} \frac{f'(x)}{x - z} d\mu_t(x) d\sigma_t(z) \right), \\ \frac{d}{dt} \left(\int_{\mathbb{H}} h(z) d\sigma_t(z) \right) &= \int_{\mathbb{R}} \int_{\mathbb{H}} \frac{2h'(z)}{z - y} d\sigma_t(z) d\mu_t(y), \end{aligned}$$

for every $f \in C_b^2(\mathbb{R}, \mathbb{C})$ and continuously differentiable $h : \mathbb{H} \rightarrow \mathbb{C}$ with h' bounded.

(ii) Let $M_t(z) := \int_{\mathbb{R}} \frac{2}{z-x} d\mu_t(x)$. Then $g_{N,t}$ converges locally uniformly to g_t which satisfies the following system of (abstract) differential equations:

$$\begin{aligned} \frac{d}{dt} g_t(z) &= M_t(g_t), \\ \frac{\partial}{\partial t} M_t(z) &= -\frac{\partial}{\partial z} M_t(z) \cdot M_t(z) + 2\operatorname{Re} \left(\int_{\mathbb{H}} \frac{M_t(z)}{(z-g_t(w))^2} - \frac{M_t(g_t(w))}{(z-g_t(w))^2} - \frac{\frac{\partial}{\partial z} M_t(z)}{(z-g_t(w))} d\sigma(w) \right). \end{aligned}$$

Proof. First we note that $\sigma_{N,t}$ is the pushforward of $\sigma_{N,0}$ w.r.t. $g_{N,t}$, i.e.

$$\sigma_{N,t} = (g_{N,t})_*(\sigma_{N,0}). \quad (3.1)$$

A normality argument plus assumption (d) would yield the existence of $T > 0$ and a compact set $K_T \subset \mathbb{H}$ such that

$$\operatorname{supp} \sigma_{N,t} \subset K_T \text{ for all } t \in [0, T]. \quad (3.2)$$

Now let $f \in C_b^2(\mathbb{R}, \mathbb{C})$. Then

$$\begin{aligned} \frac{d}{dt} \left(\int_{\mathbb{R}} f(x) d\mu_{N,t}(x) \right) &= \frac{d}{dt} \left(\sum_{k=1}^N \frac{1}{N} f(V_{N,k}(t)) \right) \\ &= \sum_{k=1}^N \frac{1}{N} f'(V_{N,k}(t)) \cdot \left(\sum_{j \neq k} \frac{2/N}{V_{N,k}(t) - V_{N,j}(t)} + \sum_{j=1}^{M_N} \frac{\alpha_{N,j}/N}{V_{N,k}(t) - S_{N,j}(t)} + \sum_{j=1}^{M_N} \frac{\alpha_{N,j}/N}{V_{N,k}(t) - \overline{S_{N,j}(t)}} \right) \\ &= \int_{x \neq y} \frac{f'(x) - f'(y)}{x - y} d\mu_{N,t}(x) d\mu_{N,t}(y) + 2\operatorname{Re} \left(\int_{\mathbb{H}} \int_{\mathbb{R}} \frac{f'(x)}{x - z} d\mu_{N,t}(x) d\sigma_{N,t}(z) \right). \end{aligned}$$

As in the proof of Theorem 2.5, we conclude that the first term is bounded. The second one is bounded for all $t \in [0, T]$ because of (3.2) and as σ is finite.

Recall that $S_{N,j}(t) = g_{N,t}(s_{N,j})$. Thus, for any continuously differentiable $h : \mathbb{H} \rightarrow \mathbb{C}$, with h' bounded, we get

$$\begin{aligned} \frac{d}{dt} \left(\int_{\mathbb{H}} h(z) d\sigma_{N,t}(z) \right) &= \frac{d}{dt} \left(\sum_{j=1}^{M_N} \frac{\alpha_{N,j}}{N} h(S_{N,j}(t)) \right) \\ &= \sum_{j=1}^{M_N} \frac{\alpha_{N,j}}{N} h'(S_{N,j}(t)) \cdot \sum_{k=1}^N \frac{2/N}{S_{N,j}(t) - V_{N,k}(t)} = \int_{\mathbb{R}} \int_{\mathbb{H}} \frac{2h'(z)}{z - y} d\sigma_{N,t}(z) d\mu_{N,t}(y), \end{aligned}$$

which is also bounded for all $t \in [0, T]$.

As in the proof of Theorem 2.5, we conclude tightness of the sequences $\{\mu_{N,t}\}_{n \in \mathbb{N}}$ and $\{\sigma_{N,t}\}_{n \in \mathbb{N}}$. It should be noted that we do not need a condition like (c) for the convergence of $\sigma_{N,0}$, as we assumed that the support of $\sigma_{N,0}$ is contained in a compact set independent of N .

Now let μ_t be the limit of a converging subsequence $\mu_{N_k,t}$. Relation (3.1) implies that $\sigma_{N,t}$ converges to $\sigma_t := (g_t)_*(\sigma)$ as $N \rightarrow \infty$, and we obtain statement (i).

As in the proof of Corollary 2.7, we conclude that $g_{N,t}$ converges locally uniformly to g_t which satisfies

$$\frac{d}{dt} g_t = M_t(g_t).$$

Finally, we can write the first equation of (i) as

$$\frac{d}{dt} \left(\int_{\mathbb{R}} f(x) d\mu_t(x) \right) = \int_{\mathbb{R}^2} \frac{f'(x) - f'(y)}{x - y} d\mu_t(x) d\mu_t(y) + 2\operatorname{Re} \left(\int_{\mathbb{H}} \int_{\mathbb{R}} \frac{f'(x)}{x - g_t(w)} d\mu_t(x) d\sigma(w) \right).$$

For $f(x) = \frac{2}{z-x}$, $z \in \mathbb{H}$, this becomes (and we use the calculation from the proof of Corollary 2.7)

$$\begin{aligned} \frac{\partial}{\partial t} M_t(z) &= -\frac{\partial}{\partial z} M_t(z) \cdot M_t(z) + \\ &2\operatorname{Re} \left(\int_{\mathbb{H}} \int_{\mathbb{R}} \frac{2}{(z-x)(z-g_t(w))^2} - \frac{2}{(g_t(w)-x)(z-g_t(w))^2} + \frac{2}{(x-z)^2(z-g_t(w))} d\mu_t(x) d\sigma(w) \right) \\ &= -\frac{\partial}{\partial z} M_t(z) \cdot M_t(z) + 2\operatorname{Re} \left(\int_{\mathbb{H}} \frac{M_t(z)}{(z-g_t(w))^2} - \frac{M_t(g_t(w))}{(z-g_t(w))^2} - \frac{\frac{\partial}{\partial z} M_t(z)}{(z-g_t(w))} d\sigma(w) \right) \end{aligned}$$

and we are done. \square

Figure 6 shows a stream plot of the trajectories for $Q(z) = \prod_{k=0}^9 (z - 2k/9 + 1)^2 \cdot (z - i)^{-10} \cdot (z + i)^{-10}$, and in Figure 7, $z = i$ is a zero of Q , i.e. $Q(z) = \prod_{k=0}^9 (z - 2k/9 + 1)^2 \cdot (z - i)^{10} \cdot (z + i)^{10}$.

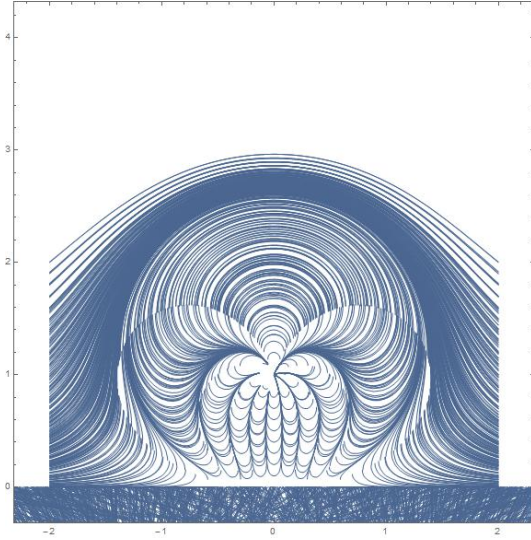


Figure 6: Pole of order N at $z = i$.

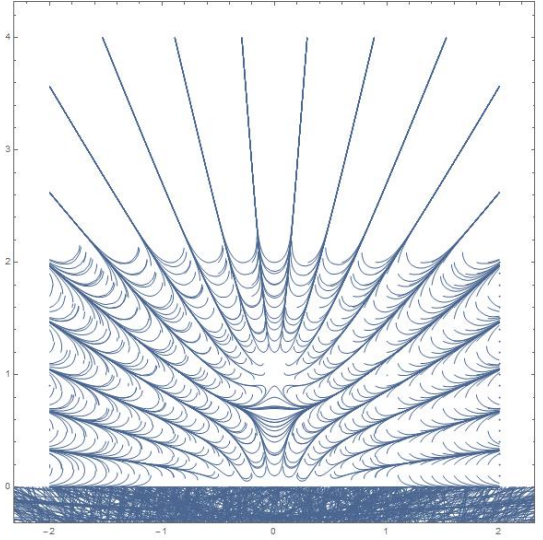


Figure 7: Zero of order N at $z = i$.

References

- [AEPA09] O. Arizmendi and V. Pérez-Abreu, *The S -transform of symmetric probability measures with unbounded supports*, Proc. Amer. Math. Soc. **137** (2009), no. 9, 3057–3066.
- [Bau04] R. O. Bauer, *Löwner’s equation from a noncommutative probability perspective*, J. Theoret. Probab. **17** (2004), no. 2, 435–456.
- [BBCL99] A. Bonami, F. Bouchut, E. Cépa, and D. Lépingle, *A nonlinear stochastic differential equation involving the Hilbert transform*, J. Funct. Anal. **165** (1999), no. 2, 390–406.
- [BBK05] M. Bauer, D. Bernard, and K. Kytölä, *Multiple Schramm-Loewner evolutions and statistical mechanics martingales*, J. Stat. Phys. **120** (2005), no. 5-6, 1125–1163.
- [Bil99] P. Billingsley, *Convergence of probability measures*, second ed., Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons, Inc., New York, 1999, A Wiley-Interscience Publication.
- [BV93] H. Bercovici and D. Voiculescu, *Free convolution of measures with unbounded support*, Indiana Univ. Math. J. **42** (1993), no. 3, 733–773.
- [Boe15] C. Boehm: *Loewner equations in multiply connected domains*, PhD thesis, Wuerzburg (2015),
- [Car03] J. Cardy, *Stochastic Loewner evolution and Dyson’s circular ensembles*, J. Phys. A **36** (2003), no. 24, L379–L386.
- [Cha92] T. Chan, *The Wigner semi-circle law and eigenvalues of matrix-valued diffusions*, Probab. Theory Related Fields **93** (1992), no. 2, 249–272.
- [CL97] E. Cépa and D. Lépingle, *Diffusing particles with electrostatic repulsion*, Probab. Theory Related Fields **107** (1997), no. 4, 429–449.
- [CL01] ———, *Brownian particles with electrostatic repulsion on the circle: Dyson’s model for unitary random matrices revisited*, ESAIM Probab. Statist. **5** (2001), 203–224 (electronic).
- [dMG16] A. del Monaco and P. Gumenyuk, *Chordal Loewner equation*, Complex analysis and dynamical systems VI. Part 2, Contemp. Math. (2016).
- [dMS16] A. del Monaco and S. Schleißinger, *Multiple SLE and the complex Burgers equation*, Mathematische Nachrichten, to appear (2016).
- [Dub07] J. Dubédat, *Commutation relations for Schramm-Loewner evolutions*, Comm. Pure Appl. Math. **60** (2007), no. 12, 1792–1847.

- [FK15a] S. M. Flores and P. Kleban, *A solution space for a system of null-state partial differential equations: Part 1*, Comm. Math. Phys. **333** (2015), no. 1, 389–434.
- [FK15b] ———, *A solution space for a system of null-state partial differential equations: Part 2*, Comm. Math. Phys. **333** (2015), no. 1, 435–481.
- [FK15c] ———, *A solution space for a system of null-state partial differential equations: Part 3*, Comm. Math. Phys. **333** (2015), no. 2, 597–667.
- [FK15d] ———, *A solution space for a system of null-state partial differential equations: Part 4*, Comm. Math. Phys. **333** (2015), no. 2, 669–715.
- [Gär88] J. Gärtner, *On the McKean-Vlasov limit for interacting diffusions*, Math. Nachr. **137** (1988), 197–248.
- [GB92] V. V. Goryainov and I. Ba, *Semigroup of conformal mappings of the upper half-plane into itself with hydrodynamic normalization at infinity*, Ukrain. Mat. Zh. **44** (1992), no. 10, 1320–1329.
- [Gra07] K. Graham, *On multiple Schramm–Loewner evolutions*, Journal of Statistical Mechanics: Theory and Experiment **2007** (2007), no. 03.
- [JVST12] F. Johansson Viklund, A. Sola, and A. Turner, *Scaling limits of anisotropic Hastings-Levitov clusters*, Ann. Inst. Henri Poincaré Probab. Stat. **48** (2012), no. 1, 235–257.
- [KL07] M. J. Kozdron and G. F. Lawler, *The configurational measure on mutually avoiding SLE paths*, Universality and renormalization, Fields Inst. Commun., vol. 50, Amer. Math. Soc., Providence, RI, 2007, pp. 199–224.
- [Koz09] M. J. Kozdron, *Using the Schramm–Loewner evolution to explain certain non-local observables in the 2D critical Ising model*, J. Phys. A **42** (2009), no. 26.
- [KP15] K. Kytölä and E. Peltola, *Pure partition functions of multiple SLEs*, arXiv:1506.02476 (2015).
- [Law05] G. F. Lawler, *Conformally invariant processes in the plane*, Mathematical Surveys and Monographs, vol. 114, American Mathematical Society, Providence, RI, 2005.
- [Law11] G. F. Lawler, *Defining SLE in multiply connected domains with the Brownian loop measure*, arXiv:1108.4364 (2011).
- [LL80] G. S. Ladde and V. Lakshmikantham, *Stochastic differential inequalities of Itô type*, Applied stochastic processes (Proc. Conf., Univ. Georgia, Athens, Ga., 1978), Academic Press, New York-London, 1980, pp. 109–120.
- [MM03] J.-F. Marckert and A. Mokkadem, *The depth first processes of Galton-Watson trees converge to the same Brownian excursion*, Ann. Probab. **31** (2003), no. 3, 1655–1678.
- [MS13] J. Miller and S. Sheffield, *Quantum Loewner Evolution*, arxiv:1312.5745v1 (2013).
- [Pom75] C. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.
- [Ric09] C. Richard, *On q -functional equations and excursion moments*, Discrete Math. **309** (2009), no. 1, 207–230.
- [RS93] L. C. G. Rogers and Z. Shi, *Interacting Brownian particles and the Wigner law*, Probab. Theory Related Fields **95** (1993), no. 4, 555–570.
- [Sch12] K. Schmüdgen, *Unbounded self-adjoint operators on Hilbert space*, Graduate Texts in Mathematics, vol. 265, Springer, Dordrecht, 2012.
- [Sch16] S. Schleißinger, *The Chordal Loewner Equation and Monotone Probability Theory*, arXiv:1605.06689 (2016).
- [Tsa09] J. Tsai, *The Loewner driving function of trajectory arcs of quadratic differentials*, J. Math. Anal. Appl. **360** (2009), no. 2, 561–576.

Andrea del Monaco: Università di Roma “Tor Vergata”, 00133 Roma, Italy.

Ikkei Hotta: Yamaguchi University, Ube 755-8611, Japan.

Sebastian Schleißinger: Università di Roma “Tor Vergata”, 00133 Roma, Italy.